Lagrangian flows within reflecting internal waves at a horizontal free-slip surface

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In this paper sequel to Zhou and Diamessis (“Reflection of an internal gravity wave beam off a horizontal free-slip surface,” Phys. Fluids \textbf{25}, 036601 (2013)), we consider Lagrangian flows within nonlinear internal waves (IWs) reflecting off a horizontal free-slip rigid lid, the latter being a model of the ocean surface. The problem is approached both analytically using small-amplitude approximations and numerically by tracking Lagrangian fluid particles in direct numerical simulation (DNS) datasets of the Eulerian flow. Inviscid small-amplitude analyses for both plane IWs and IW beams (IWBs) show that Eulerian mean flow due to wave-wave interaction and wave-induced Stokes drift cancels each other out completely at the second order in wave steepness $A$, i.e., $O(A^2)$, implying zero Lagrangian mean flow up to that order. However, high-accuracy particle tracking in finite-Reynolds-number fully nonlinear DNS datasets from the work of Zhou and Diamessis suggests that the Euler-Stokes cancelation on $O(A^2)$ is not complete. This partial cancelation significantly weakens the mean Lagrangian flows but does not entirely eliminate them. As a result, reflecting nonlinear IWBs produce mean Lagrangian drifts on $O(A^2)$ and thus particle dispersion on $O(A^4)$. The above findings can be relevant to predicting IW-driven mass transport in the oceanic surface and subsurface region which bears important observational and environmental implications, under circumstances where the effect of Earth rotation can be ignored. © 2015 AIP Publishing LLC.

I. INTRODUCTION

Internal waves (IWs) in the oceanic environment are usually of tidal origin\textsuperscript{1} and often observed in the form of internal wave beams (IWBs).\textsuperscript{2–4} These tidal wave beams bear important implications for ocean energetics and mixing.\textsuperscript{5} Waves of such form can also be generated by a stratified turbulent flow,\textsuperscript{6} and these waves of turbulent origin can possibly reveal their submerged sources on remote sensing images of the ocean surface.\textsuperscript{7,8}

Specifically, we are interested in IWBs which propagate upward away from their sources and towards the upper layer/boundary of the ocean. In a two-dimensional setup, these upward-going IWBs are subject to linear or nonlinear processes, such as refraction and reflection when stratification is variable in the vertical,\textsuperscript{9–13} tunneling when encountering a critical layer,\textsuperscript{14} and reflection when reaching the sea surface.\textsuperscript{15,16} Parametric subharmonic instability (PSI) in two-dimensional IWBs has recently been treated by various papers.\textsuperscript{16–20} When IWBs become three-dimensional, they may drive mean flows either due to modulational instability\textsuperscript{21,22} or a dissipative mechanism.\textsuperscript{23}

In a recent paper by Zhou and Diamessis\textsuperscript{16} (hereinafter referred to as Z&D), we focused on the nonlinear reflection process of a two-dimensional IWB off a horizontal free-slip rigid lid which is a model of the sea surface. Mean currents form beneath the free-slip reflecting surface due to wave-wave interaction between incident and reflected beams. The magnitude of these Eulerian

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mean flows observed in our direct numerical simulation (DNS) agrees very well with the predictions of a weakly nonlinear theory developed by Tabaei, Akylas, and Lamb24 (hereinafter referred to as TAL).

In this paper, we aim to extend Z&D from the Eulerian to the Lagrangian perspective with the motivation explained below. First, net Lagrangian effects such as mean drifts and particle dispersion due to IWB reflection have a direct impact on the transport of passive scalars and microorganisms in the ocean upper layer. However, knowledge of the Eulerian mean of a wavy flow alone is often found to be insufficient for inferring material transport when wave-induced effects, such as Stokes drift, are significant, e.g., in the context of interpreting field data containing both surface waves and mean currents.25 This gives rise to a need to further clarify the nonlinear reflection process from the Lagrangian perspective. Moreover, the understanding of reflecting-IWB-induced Lagrangian motions of ocean surface tracers may facilitate the interpretation of remote sensing images containing possible signatures of IWBs and eventually the identification of submerged IWB sources.7,8

Lagrangian studies of IWs have been reported in the literature targeting wave configurations other than reflecting IWBs: Stokes drift induced by vertically localized IW packets can be derived from the Stokes theorem.26 Shear-flow dispersion due to vertically unbounded IWs has been examined by numerical particle tracking.27 Lagrangian flows within vertical-mode IWs and internal solitary waves have been investigated by Gil and Fringer.31 Particle dispersion due to dissipative effects in a random IW field has also been investigated.28 In this paper, we examine the Lagrangian aspects of reflecting IWs off a free-slip surface aiming at the following specific questions: Does a reflecting finite-amplitude IWB induce net Lagrangian effects, i.e., mean drifts and dispersion? If so, how do these effects scale with IWB parameters? Both small-amplitude analysis (Sec. II) and fully nonlinear numerical particle tracking (Sec. III) in the Z&D DNS dataset will be utilized to address these questions. In Sec. IV, further discussion of the results will be provided. Unless otherwise specified, the nomenclature in this paper will follow that of Z&D.

II. SMALL-AMPLITUDE ANALYSIS

In this section, results on two-dimensional small-amplitude analysis of Lagrangian flow within reflecting IWs are discussed. Readers who are interested in the details of the derivation are referred to Appendices A and B. Of particular interest is the wave-mean Lagrangian motion of fluid particles, which can be decomposed into the Eulerian mean flow ⟨u⟩ and the Stokes drift uS, as shown in Eq. (A8). One may define the wave amplitude parameter A, or steepness A, as

\[ A \equiv \frac{U_0}{c_s} = \frac{U_0 k_x}{\omega} = \frac{U_0 k_x}{N \cos \theta} = \frac{2\pi U_0}{N \lambda_x \cos \theta}, \]

(1)

which is the ratio of the maximum horizontal velocity U0 of a fluid particle in a freely propagating wave, to the horizontal phase speed cs of the wave. At the second-order in the wave steepness A, i.e., O(A2), the mean Lagrangian drift ⟨uL⟩ consists of the Eulerian mean flow ⟨u⟩ and the O(A2) Stokes drift uS.30 the latter exclusively driven by the O(A) reflecting waves (see the scaling arguments in Appendix A). The analyses shown in Appendix B suggest that, for an inviscid reflecting plane IW off a rigid horizontal surface, ⟨u⟩ and uS cancel each other out completely everywhere in the flow, driving no net Lagrangian transport up to the second order. Interestingly, a similar observation of local Euler-Stokes cancelation has also been reported for traveling surface waves in the laboratory31 and in the field.25

An explicit derivation of Euler-Stokes mean flow cancelation for a reflecting plane internal wave, following the formalism developed by Longuet-Higgins32,33 which has been applied to other IW configurations.34,35 may be found in Appendix B. To this end, and in a more general wave framework, the classical generalized Lagrangian mean (GLM) theory developed by Andrews and McIntyre36 (see specific applications of the GLM theory to IWs in the work of Grimshaw37) may also be straightforwardly applied to the inviscid formulation of the problem under consideration in this paper to predict the nonexistence of Lagrangian mean flow therein. Andrews and McIntyre suggest that the Lagrangian mean flows arise due to the transient effects on the wave field, viscous
damping, forcing, or specific boundary conditions. Without any of these factors present, the inviscid plane IW reflection is not expected to drive any Lagrangian mean flows.

Thorpe,\textsuperscript{38,39} in his studies of plane IW reflection off a slope bottom boundary, anticipated that the mass transport within the inviscid waves is zero, without showing an explicit derivation. According to Thorpe\textsuperscript{38} (pg. 283), no Lagrangian mean flow is expected within a plane IW reflecting upon a solid slope, “as it must to preserve the density field” in a nondiffusive setup. This argument is appropriate for the case of a sloping boundary, as any net mass transport in the along-slope direction would involve permanent perturbation to the density field and is therefore expected to vanish. However, the same reasoning might not be directly applicable to our case with a horizontal reflecting surface, as the mean Lagrangian motion in the horizontal direction (parallel to the reflecting surface) does not induce diapycnal mass transport and thus cannot be ruled out simply by Thorpe’s argument based on the preservation of the density field.

The $O(A^2)$ analysis for a plane IW can be readily extended to an IWB. A sample calculation is shown here for an IWB of the same geometry as the ones studied in simulations T01–05 of Z&D, i.e., IWBS of $\theta = 45^\circ$ and surface-observed compactness $\lambda_s/\sigma_{sfc} = 1/0.861$, where $\sigma_{sfc}$ is the half-width of the wave envelope at the surface along the $x$ axis (see the DNS setup in Fig. 2). Specifically, the wave kinematics, including the primary wave $(u,w)_I$ and the mean flow $\langle u \rangle$, can be obtained by applying the TAL theory.\textsuperscript{24} Eqs. (B3)–(B8) can be computed numerically for any IWB with known characteristics to obtain the second-order Stokes drift.

The results of such second-order analyses on Eulerian mean flow and Stokes drift due to a reflecting beam are shown in Fig. 1. In contrast to the plane IW reflection case where the waves fill out the entire domain and produce nonlinear effects (either Eulerian mean or Stokes) everywhere, as suggested by Eqs. (B1) and (B9), the incident and reflected IWBS produce $O(A^2)$ effects only where the two beams overlap beneath the reflecting surface. Other than the spatial confinement of any nonlinear effect, there is no substantial difference between the plane IW and IWB cases in terms of Lagrangian mean flows of $O(A^2)$: The Eulerian mean flow $\langle u \rangle$ in Fig. 1(a) and second-order Stokes drift $u_{S,II}$ in Fig. 1(b) have exactly the same structure but opposite signs everywhere in the flow. Therefore, due to the complete cancelation of Eulerian mean and Stokes drift, one would expect no net $O(A^2)$ Lagrangian effects within an inviscid weakly nonlinear reflecting IW. However, when the fluid does become viscous, the cancelation is not expected to hold; in the context of the GLM theory,\textsuperscript{36} the only plausible mechanism to drive Lagrangian mean flow in the current setup.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{(a) Eulerian mean flow, $\langle u \rangle$, as predicted by the TAL\textsuperscript{24} solution, and (b) second-order Stokes drift, $u_{S,II}$, due to a reflecting IW of $\theta = 45^\circ$ and compactness $\lambda_s/\sigma_{sfc} = 1/0.861$. Both velocities are normalized by $2U_0^2/c_x$. $(x,z')=(0,0)$ is the point where the centerline of the incident beam hits the reflecting surface, and the incident and reflected beams overlap approximately within $-\lambda_s < x < \lambda_s$ and $-\lambda_z < z' < 0$. The two velocity fields, $\langle u \rangle$ and $u_{S,II}$, which are computed independently, have the same magnitude but opposite signs everywhere in the flow, and thus, they cancel out each other completely.}
\end{figure}
is through the spatial inhomogeneity produced by the viscous dissipation along the IW’s path of propagation. In Sec. III, results from viscous fully nonlinear computations, showing nonzero net Lagrangian mean flows, are presented.

III. NUMERICAL PARTICLE TRACKING

A. Numerical methods

In this section, we report fully nonlinear results of Lagrangian motions obtained by applying numerically particle tracking to the Z&D DNS dataset (see the simulation setup in Fig. 2). Since the net Lagrangian effects being investigated are expected to be nonlinear dynamics of high order \([O(A^2)]\), a reliable particle tracker is imperative for our study. A high-accuracy particle-tracking scheme, which is tailored to the specific Navier-Stokes solver used by Z&D, has been developed and implemented in parallel. The time-marching scheme employs a locally fourth-order Adams-Bashforth-Moulton method. A global Fourier-Lagrange interpolation scheme in two dimensions has been developed in order to interpolate the Eulerian velocity field on the Z&D grid to the instantaneous positions of the particles, i.e., Eqs. (A2) and (A3), with spectral accuracy in both spatial directions \(x\) and \(z\): For \(x\), a periodic direction in the flow solver, a Fourier interpolation technique is used. For \(z\), which is discretized by spectral multi-domains, Lagrange interpolants on the Gauss-Legendre-Lobatto grid are deployed for each individual vertical subdomain. Auxiliary tests have demonstrated that, under this global Fourier-Lagrange interpolation scheme and compared to a third-order cubic spline local one, the interpolation error can be reduced by at least three orders of magnitude at the spatial resolution used by Z&D. This high accuracy strengthens the reliability of the particle tracking results to be presented in this section.

B. Parameter space

The parameter space associated with numerical particle tracking, covering various wave configurations (see Table I) examined by Z&D and particle characteristics considered in the present study, is described below.

1. Wave configurations

The first dimension of our parameter space results from the specific wave configurations examined by Z&D which sought to cover a range of wave steepnesses \(A\), beam angles \(\theta\) (also a measure of the wave’s hydrostaticity), beam compactnesses \(\lambda_x/\sigma_{sf,c}\), and Reynolds numbers \(Re_w\). For the purpose of the Lagrangian study in this paper, waves of very small amplitudes \((A<1.6\%)\), which were examined by Z&D in the Eulerian reference frame, will not be considered. These very-small-amplitude waves may drive very weak nonlinear Lagrangian effects, the magnitudes of
TABLE I. Summary of wave configurations. θ is the angle between the group velocity vector of the IWB and the vertical direction; σ_{sfc} is the observed wave envelope half-width at the surface; Re_{w} ≡ \lambda_{w}^{2} N / ν is the wave-based Reynolds number; A ≡ U_{0} / c_{x} is the dimensionless wave amplitude (steepness).

<table>
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<th>Run no.</th>
<th>θ (deg)</th>
<th>σ_{sfc}/A</th>
<th>Re_{w}</th>
<th>A (%)</th>
<th>Run no.</th>
<th>θ (deg)</th>
<th>σ_{sfc}/A</th>
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<th>A (%)</th>
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</tbody>
</table>

which are expected to be comparable to those of the small interpolation errors of the high-accuracy numerical particle tracking.

2. Particle characteristics

A particle to be considered is defined by its initial position \((x_{0}^{+}, z_{0}^{+})\) and initial time \(t_{0}\) at which it assumes the designated initial position. To obtain a full coverage of the three-dimensional parameter space \((x_{0}^{+}, z_{0}^{+}, t_{0})\) associated with particles, the following strategy is used. A batch of particles are inserted at the same set of positions at \(q\) (typically 18) evenly spaced phases of the wave cycle respectively to cover various \(t_{0}\)'s. After the insertion, a particle is typically tracked for five wave periods. A two-dimensional array of \(p^2\) particles (typically \(p = 16\)) is inserted at each designated insertion time uniformly in space to cover one wavelength in both \(x\) and \(z\) within the core of the IWB reflection zone, i.e., on \(-\lambda_{x}/2 < x < \lambda_{x}/2\) and \(-\lambda_{z} < z < 0\), where \((x, z) = (0, 0)\) is the position at which the centerline of the incident beam intersects with the top surface. As a result, to calculate the desired Lagrangian statistics, a population of \(p^2q = 16^2 \times 18 = 4608\) particles are tracked for each wave configuration to cover a complete range of \((x_{0}^{+}, z_{0}^{+}, t_{0})\) combinations within a reflecting IWB. The particles are tracked when the nonlinear reflection region has fully developed in the simulations, i.e., there is no transient effect due to the initial impingement of the beam onto the surface. For cases where the beam will later undergo PSI, i.e., T06, T07, and T11, particle tracking is performed at earlier stage of the reflection before any effect of subharmonic waves has established.

C. Particle orbits

This section presents qualitative yet highly insightful results on the orbits followed by Lagrangian particles within a reflecting IWB. A first understanding of particle kinematics may be obtained by animating the particle trajectories with respect to \(t_{0}\), the time at which the particles are initiated at the designated locations. The animations are available in the multimedia material accompanying this paper. One salient point of these animations is that the exact positions of the particles during the wave cycles depend strongly on the initial time of tracking, although the particles are initiated at the same set of locations, which confirms the necessity of including \(t_{0}\) in our parameter space.

One can qualitatively observe differing degrees of wave-mean drifts in the particle orbits from one wave cycle to the next, for various wave configurations under consideration (see Table I).

a. When the wave steepness \(A\) is considerably small (i.e., T01–03, T16–17, and T14), particle trajectories for multiple wave periods (not shown) virtually overlap into one visible orbit for each \((x_{0}^{+}, z_{0}^{+})\) location, which means that the drift of the mean orbital position during these wave cycles is very small.

b. When the steepness \(A\) increases sufficiently beyond the value of the most linear cases considered above, the amount of drift does become significant. An example from T13 is shown in Fig. 3.
c. When \( A \) increases further, the drifts become even more significant (i.e., T11–12). An example from T12, with amplitude almost doubled from T13 (Fig. 3) (Multimedia view), is shown in Fig. 4 (Multimedia view). Due to the larger amount of drift in the particles’ mean orbital positions, the shape of the orbits of some particles undergoes noticeable changes from the previous wave cycle to the next (see the differences in the blue and red orbits in Fig. 4) (Multimedia view). Interestingly, although the particle Lagrangian motion has partially lost its periodicity in time, suggesting that the waves are of rather high amplitudes and thus highly nonlinear, these waves are fairly stable when viewed in the Eulerian reference frame.

It is also interesting to note that a beam angle \( \theta \) that is more inclined to the horizontal seems to drive more significant Lagrangian drifts; among the three beam examined, the most dramatic drifts are observed for the “shallowest” beam (greatest inclination to the horizontal and thus the most

FIG. 3. Typical particle orbits from T13 (\( A = 16.0\% \) and \( \theta = 63^\circ \)). Left panel shows the full field of view; right panel shows a zoomed-in view within the dashed-line box drawn in the left panel. Trajectories of a two-dimensional array of 16 \times 16 = 256 particles are plotted over five wave periods. A net drift can now be observed for each initial position, while the orbital shape remains repeatable from one wave cycle to the next. (Multimedia view) [URL: http://dx.doi.org/10.1063/1.4936578.1]

FIG. 4. Typical particle orbits from T12 (\( A = 33.3\% \) and \( \theta = 63^\circ \)). Particle orbits are shown for two wave cycles only: Trajectories during the first cycle are drawn in blue, and the immediate following cycle in red. Particles are downsampled in the vertical for clarity. The orbital shape of some particles varies from the previous wave cycle to the next, and a great amount of net drift in the mean orbital position can be observed. (Multimedia view) [URL: http://dx.doi.org/10.1063/1.4936578.2]
hydrostatic\textsuperscript{26}, i.e., with $\theta = 63^\circ$. Parameterization of Lagrangian drifts as a function of $A$ (which itself is also a function of $\theta$) will be discussed in Subsection III D.

D. Mean Lagrangian drifts and dispersion

1. Drifts

a. Drift direction and particle dispersion. Here the wave-mean Lagrangian drift or mass transport velocity $\langle u^+ \rangle$ is considered. Of interest is the direction of $\langle u^+ \rangle$ as one scans vertically through the water column. This can be observed visually by comparing the particles’ horizontal positions before and after the wave effects, and a typical plot is shown in Fig. 5 for T13. Driven by the reflecting IW beam, particles disperse horizontally around the $x$ position at which they are inserted; the depth-averaged mean position of the particles does not seem to be shifted significantly. The degree of the horizontal dispersion seems to grow with $x$, suggesting some spatial inhomogeneity of the particle drift statistics in the horizontal direction.

b. Horizontal profiles. Here, the statistics of wave-mean drift $\langle u^+ \rangle$ is considered. The results reported here are obtained by averaging over five wave periods to minimize the statistical uncertainty of averaging over one period only as in the standard definition of the operator $\langle \cdot \rangle$ in Eq. (A6). Our specific goal here is to examine how the depth-averaged drifts vary in $x$. The particles tracked have covered the three-dimensional parameter space $(x_0^+ z_0^+, t_0)$; for each $x_0^+$, averages are taken over all $p z_0^+$ locations, covering one wavelength $\lambda_z$, and all $q$ insertion time $t_0$'s, covering one wave period. Both the arithmetic mean and the quadratic mean (rms) based on $pq = 16 \times 18 = 288$ particles at each $x_0^+$ are considered.

Typical horizontal profiles of these mean drifts can be found in Fig. 6 for T11–14. These runs are chosen to show various degrees of particle drifts at the same beam angle $\theta = 63^\circ$ (Table I). First, for all cases, the rms mean is one order of magnitude bigger than the arithmetic mean, which means that the variations in $\langle u^+ \rangle$ at each horizontal location in the water column are much greater than the depth average of them. This implies that the reflecting IWB is much more effective in dispersing the particles in opposite directions horizontally throughout the water column than in drifting the particles towards a preferred direction.

Second, for T12–14, the drifts increase monotonically with $x$ in $-0.5 \lambda_x < x < 0.5 \lambda_x$, which suggests that the maximum drift magnitude may occur at $x > 0$, i.e., where the beam centerline has already deflected. The exception to this observation is T11, the most nonlinear case among all, where the drift is almost constant in $x$. It is somewhat unexpected to see such dependence of Lagrangian statistics in $x$ over one wavelength $\lambda_x$; however, note that $x$ is not a statistically homogeneous direction in the IWB reflection setup — the wave does vary in $x$ due to the structural

![Fig. 5](file_url)

**FIG. 5.** Particle positions $(x_0^+, z_0^+)$ at (a) the beginning and (b) the end of five wave periods (T13). Marked grey in (b) is the initial $x$ positions at which the particles are inserted. Particles shown are inserted at the same phase of the wave.
modulations imposed by the wave envelope, as well as due to the viscous decay of wave amplitude along the path of the wave’s (energy) propagation.

Finally, the magnitude of the drift clearly increases with wave steepness $A$, e.g., for T11, the rms drift exceeds 5% of phase speed $c_x$, which implies a considerable amount of wave-induced mass transport; the scaling of the drifts with wave parameters for all cases is discussed next.

c. Scaling. Fig. 7 shows the Lagrangian mean drift velocity scale $U_L$, taken as the maximum rms $\langle u^+ \rangle$ at all $x^+_0$’s for each wave configuration, plotted against $A$, a parameter which has successfully scaled the Eulerian mean flow in the work of Z&D and is consistent with the TAL theory.

First, the plot clearly suggests a quadratic dependence of $U_L$ on $A$, although small-amplitude analysis in Sec. II shows that the net Lagrangian effect is zero to $O(A^2)$. The same dependence on $A$ also
applies to all cases, despite the differences in their qualitative features. Moreover, given the fact that the quadratic scaling shown in Fig. 7, i.e., $U_L/c_x \propto A^2$, collapses the data points from different $\theta$'s quite well, it is very likely the same $A$ parameter (which has scaled the Eulerian mean successfully) can also be applicable to scale the mean Lagrangian flows, in addition to scaling the Eulerian mean. However, as evidenced by the vertical offset of almost one decade between the two different scalings in Fig. 7, the magnitude of rms Lagrangian drift $U_L$ is significantly smaller than its counterpart $U_E$, the vertical rms velocity of the Eulerian mean flows, at a given $A$.

2. Dispersion

As mentioned earlier, the net Lagrangian effect due to reflecting IW (at a given $x$ position) is mainly a horizontal dispersion rather than driving an unidirectional drift. Here, one can further quantify the wave-induced particle dispersion by scaling the horizontal dispersion coefficient, $K_x$, as in

$$\sigma^2 = 2K_x \Delta t,$$  \hspace{1cm} (2)

where $\sigma^2$ is the variance of particle displacements after time $\Delta t$, which, in the context of a wave, can be taken as one or multiple of the wave period(s). Based on previous results on the scaling of the drift velocity scale $U_L \sim O(A^2)$, one would expect that

$$K_x = \frac{\sigma^2}{2\Delta t} \sim \frac{(U_LT)^2}{T} \sim U_L^2 T \propto A^4.$$  \hspace{1cm} (3)

Here the expectation that $K_x$ is of $O(A^4)$ is verified by examining the particle tracking results. First, the variance of the net displacements of $q$ particles initiated at a given $(x_0^+,z_0^+)$ position after five periods can be calculated; then, an ensemble average of the pointwise $\sigma^2$'s based on $p^2$ initial locations can be computed for each wave configuration; and finally, $K_x$ can be estimated by Eq. (2) with $\Delta t = 5T$. The dispersion coefficients so computed are shown in Fig. 8. A clear quartic scaling of $K_x$ against $A$ can be observed, which is expected from Eq. (3). Again, the quartic scaling of dispersion is a direct implication of the wave-mean Lagrangian mean flow $U_L$ being $O(A^2)$, leading the dispersivity $K_x$, which is quadratic in $U_L$ [see Eq. (3)], to take effect on $O(A^4)$.

IV. DISCUSSION

A. Analysis vs. numerical particle tracking

The discussion in Sec. II indicates that the Eulerian mean flow due to wave-wave interaction and wave-induced Stokes drift cancel each other out to $O(A^2)$, as a consequence of the GLM theory applied to inviscid waves; in Sec. III, numerical particle tracking reveals the existence

FIG. 8. Quartic scaling $K_x/(\Lambda_x^2/T) \propto A^4$ of dispersion coefficient $K_x$ within a reflecting IWB. Dashed line represents power law $\propto A^4$. 

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of $O(A^2)$ net drifts in the viscous DNS. According to the GLM theory, as no transient effect or forcing is present, the only possible mechanism for the observed Lagrangian mean flow in the DNS considered here (Sec. III) is the viscous attenuation undergone by the IWs. Such a generation mechanism for the Lagrangian mean flow can find its analogy in acoustics as first studied by Lighthill:43,44 When sound waves are attenuated viscously, gradients of Reynolds stresses emerge, which drive mean Lagrangian drifts known as acoustic streaming or sonic wind. The amplitude of the viscous-driven steady streaming is proportional to the square of the sound wave amplitude, which is analogous to the $O(A^2)$ net drifts observed in the present study for IW reflection and reported by Bühler et al., for random IW field.28

Another interesting fact is that at a higher value of $\theta$, the higher harmonics generated upon beam reflection may become a propagating mode which can be considerably energetic.24 For example, in cases 11–14, the second harmonic is indeed present and appears on $O(A^2)$.16 A natural speculation might be that the higher harmonics may cause the stronger drifts that are observed for these cases. However, this mechanism is not supported by the scaling arguments. Recall that Stokes drift due to $O(A^n)$ dynamics can take place at best at $O(A^{n+1})$ (see in Appendix A); specifically, the $O(A^2)$ harmonic with frequency $2\omega$ is expected to generate net Lagrangian effects at $O(A^3)$ or higher. Since the drifts are observed to be on $O(A^2)$ for all cases including 11–14, it cannot be exclusively a harmonic effect.

Bühler et al.,28 convected particles in a three-dimensional near-inertial wave field and reported stronger dispersion at second order in wave amplitude, contrary to the fourth-order dispersion reported in this study. This stronger dispersion observed was attributed to weak damping of the waves. The discrepancy in the observed diffusivity (or dispersion coefficient) might be due to the inherent difference in the wave setups, i.e., Bühler et al. studied random waves at near-inertial scales forced by white noise, whereas the present study concerns a deterministic quasi-monochromatic internal wave beam free of rotation. It is worth noting that, despite these differences in the setups, the driving mechanism of the observed particle dispersion among the two studies is indeed a common one, i.e., the viscous damping undergone by these waves.

B. Open questions

A relevant topic of discussion is the possible extensions of this paper to realistic oceanic conditions. In the work of Z&D16 and this paper, a uniform stratification in the water column and a free-slip surface at the upper boundary are considered. This combination may, at first glance, seem to be a simplistic configuration, but it serves as a very important first step towards exploring more complex and realistic conditions. First, uniform stratification allows one to isolate the effects of nonlinear wave reflection, a canonical scenario of wave-wave interaction; the extension of this Lagrangian study to more realistic oceanic setups taking into account the various dynamics due to IW interaction with a variable environment (stratification profile and background flows) certainly deserves further research efforts. Second, it was found out by Z&D that a free-slip boundary acts very similarly to the base of a mixed layer in terms of the dynamics of mean flow generation — the applicability of the results shown here can in fact be much broader than the sole case of reflection off the sea surface.

Extension of the results from a free-slip reflecting surface to a no-slip one may require some extra caution. Laboratory measurements of mean flows within reflecting beams off a no-slip solid slope45,46 suggest qualitatively different results of the mean-flow structure than those reported in the work of Z&D, which is expected to be a result of the three-dimensional configuration of the experiments.52 Rodenborn et al.47 also reported discrepancy with the TAL theory24 in terms of harmonic generation at a solid bottom slope. It is to note that both leading-order wave [Eq. (B2)] and Eulerian mean [Eq. (B1)] in the inviscid solution satisfy the free-slip boundary condition automatically, which makes the small-amplitude solution directly applicable to the analytical inviscid treatment of our problem (Sec. II), but not the one with a no-slip surface. Under the no-slip condition, viscous/dissipative effects are expected to be even more important near the wall, adding another layer of complexity. Therefore, the formulation of these inviscid theories needs to be adapted for the
no-slip configurations. How the reflection-induced Lagrangian dynamics may be altered in such a highly dissipative near-bottom region remains an open question.

Finally, only two-dimensional and stable reflecting waves with no transient effects are considered in this paper. Highly intriguing areas of future investigations include Lagrangian studies in laterally confined three-dimensional wave configurations and for IWs undergoing gradually growing instabilities such as PSI. Moreover, oceanic IWBs of tidal origins tend to have fairly large values of $\theta$, i.e., $\theta > 85^\circ$, and thus high hydrostaticity. These relatively low-frequency IWBs may also be subjected to the effect of the earth’s rotation, which needs to be accounted for in a complete analysis. In addition, oceanic tidal IWBs of shallow angles may generate more energetic propagating higher harmonics that may induce significant net Lagrangian effects of their own, which could be another important difference in the realistic oceanic situation.

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APPENDIX A: SCALING STOKES DRIFT

Consider a fluid particle with an initial/reference position $(x_0^+, z_0^+)$ at $t = t_0$. Note that the superscript $^+$ will be used to denote quantities associated with the Lagrangian frame. The instantaneous position of this particle at time $t$ is given exactly as

$$(x^+, z^+) = (x_0^+, z_0^+) + \int_{t_0}^{t} u^+(t')dz', w^+(t')dt',$$

where

$$u^+(t') \equiv u(x^+(t'), z^+(t'), t'),$$
$$w^+(t') \equiv w(x^+(t'), z^+(t'), t'),$$

and $(u, w)$ are the Eulerian velocities. For small particle displacements, one can approximate the instantaneous velocity $u^+(t)$ by a Taylor series expansion in space around $(x_0^+, z_0^+)$,

$$u^+(t) = u(x_0^+, z_0^+, t) + \frac{\partial u}{\partial x}\bigg|_{x_0^+, z_0^+, t} \Delta x^+(t) + \frac{\partial u}{\partial z}\bigg|_{x_0^+, z_0^+, t} \Delta z^+(t) + \text{h.o.t.},$$

where the displacement

$$(\Delta x^+, \Delta z^+) \equiv (x^+, z^+) - (x_0^+, z_0^+).$$

One can also define the temporal wave average of any quantity $f$ as

$$\langle f \rangle \equiv \frac{1}{T} \int_{t_0}^{t_0+T} f(t')dt'.$$
with $T$ being the wave period, and apply this wave average to Eq. (A4) to obtain an expression for the wave-mean Lagrangian velocity $\langle u^+ \rangle$ for a particle initially at $(x_0^+,z_0^+)$ at the beginning of the wave cycle, i.e., $t = t_0$,

$$\langle u^+ \rangle = \langle u \rangle + \langle \frac{\partial u}{\partial x} \Delta x^+ \rangle + \langle \frac{\partial u}{\partial z} \Delta z^+ \rangle + \text{h.o.t.},$$

(A7)

where the first term on the RHS, i.e., $\langle u \rangle$, corresponds to the mean Eulerian velocity at their reference point $(x_0^+,z_0^+)$, and all the other terms on the RHS sum up to the Stokes drift $u_S$. One can symbolically rewrite Eq. (A7) as

$$\langle \text{Lagrange} \rangle = \langle \text{Euler} \rangle + \text{Stokes},$$

(A8)

which is a canonical decomposition for wave-mean Lagrangian motions in wavy flows.\textsuperscript{25,33,49}

Examining the Stokes drift terms in Eq. (A7) more closely, one may define

$$u_S \equiv \langle \frac{\partial u}{\partial x} \Delta x^+ \rangle + \langle \frac{\partial u}{\partial z} \Delta z^+ \rangle + \text{h.o.t.}$$

(A9)

The two leading terms of $u_S$ can be considered as the operator

$$\langle \frac{\partial}{\partial x} \Delta x^+ \rangle + \langle \frac{\partial}{\partial z} \Delta z^+ \rangle$$

(A10)

which is applied to $u$. Following the scaling used in Sec. II of Z&D,\textsuperscript{16} i.e.,

$$\frac{\partial}{\partial x} \sim \frac{\partial}{\partial z} \sim \frac{1}{\lambda_x}, \Delta x^+ \sim \Delta z^+ \sim U_0 T, \langle \color{white}{\frac{1}{\lambda_x}} \rangle \sim \frac{1}{T} \int_{t_0 \Delta x}^t dt \sim 1,$$

(A11)

the leading-order Stokes operator, i.e., (A10), scales linearly in wave steepness $A$,

$$\langle \frac{\partial}{\partial x} \Delta x^+ \rangle \sim \langle \frac{\partial}{\partial z} \Delta z^+ \rangle \sim \frac{U_0 T}{\lambda_x} \sim O(A).$$

(A12)

This Stokes operator, $O(A)$ by itself, is to be applied to various orders of dynamics in $u$ to produce the corresponding leading-order Stokes drift due to each order of $u$, respectively. Equation (A12) bears the simple but non-trivial implication that, for the $O(A^n)$ dynamics of $u$, the corresponding leading-order Stokes drift due to $u$ of this order is, at most, produced on the immediate higher order, i.e., $O(A^{n+1})$.

For the IW reflection problem, both the weakly nonlinear theory\textsuperscript{24,38} and DNS by Z&D suggest that along with the primary reflecting wave of leading-order $O(A)$, an Eulerian mean flow and a second harmonic coexist on the second order $O(A^2)$. The $O(A)$ primary wave, consisting of both incident and reflected waves, is expected, based on the aforementioned scaling, to produce a Stokes drift on $O(A^2)$; the $O(A^2)$ second harmonic drives a Stokes drift which is at best $O(A^3)$ and will be excluded from the scope of an $O(A^3)$ analysis. Therefore, to obtain an approximation for mean Lagrangian flow $\langle u^+ \rangle$ up to $O(A^3)$, i.e., $\langle u_L, u^+ \rangle$,\textsuperscript{30} one would need to take into account both the $O(A^2)$ Eulerian mean $\langle u \rangle$ and Stokes drift $u_{S,H}$, the latter being exclusively driven by the $O(A)$ primary wave. Recalling Eq. (A7), one can write the $O(A^2)$ balance

$$\langle u_{L,H} \rangle = \langle u \rangle + u_{S,H},$$

(A13)

where $u_{S,H}$ can be obtained by applying Stokes operator (A10) to $u$ velocities due to the $O(A)$ primary wave, i.e., $u_I$.

**APPENDIX B: EULER-STOKES CANCELLATION FOR PLANE IW REFLECTION**

Eq. (A13) now guides the derivation of an $O(A^2)$ approximation to the mean Lagrangian flows due to a plane IW reflection.
1. Eulerian mean flow

For the reflection of a plane IW, a semi-infinite domain, where \(-\infty < x < \infty \) and \(-\infty < z' \leq 0\), is considered. The reflecting surface is located at \(z' = 0\), and the incident wave has an upward (positive) group velocity at an angle \(\theta\) with respect to \(z'\). According to the inviscid weakly nonlinear solution\(^{24,38}\) for reflections off a rigid lid, the Eulerian mean flow due to wave-wave interaction in reflecting IW at a horizontal boundary is (see the derivation in the work of Z&D, Appendix A)

\[
\langle u \rangle = -\frac{2U_0^2}{c_x} \cos(2k_z z').
\]

(B1)

The Eulerian mean flow \(\langle u \rangle\) is constant in \(x\) and periodic in \(z'\) with a wavelength equal to half of the vertical wavelength \(2\pi/k_z\) of the incident (or reflected) wave.

2. Stokes drift

To obtain the \(O(A^2)\) Stokes drift \(u_{S,lt}\), one is to apply the Stokes operator in (A10) to the inviscid leading-order (linear) solution\(^{30}\) for plane IW reflection. The solution contains the contribution from the incident wave

\[
(u, w)_{inc, t} = [U_0 \sin(k_x x - k_z z' - \omega t + \alpha), U_0 \cot \theta \sin(k_x x - k_z z' - \omega t + \alpha)]
\]

and the reflected wave

\[
(u, w)_{refl, t} = [U_0 \sin(k_x x + k_z z' - \omega t + \alpha), -U_0 \cot \theta \sin(k_x x + k_z z' - \omega t + \alpha)],
\]

respectively, where \(\alpha\) is an arbitrary phase lag. Note that the dimensionless wave amplitude (steepness) \(A\) can be computed based on the maximum horizontal velocity \(U_0\) within each of the leading-order waves [see Eq. (1)]. Summing the two linear waves up results in

\[
(u, w)_t = (u, w)_{inc, t} + (u, w)_{refl, t}
\]

(B2)

\[
= (2U_0 \sin \phi \cos(k_z z'), -2U_0 \cot \theta \cos \phi \sin(k_z z')),
\]

where \(\phi(x, t) = k_x x - \omega t + \alpha\).

Consider a particle with initial position \((x, z') = (x_0^+, z_0^+)\) at \(t = t_0\). The leading-order, i.e., \(O(A)\), approximation to its instantaneous displacement can be obtained by integrating the velocities \((u, w)_t\) at the initial position,\(^{33,48}\)

\[
\Delta x^+ \approx \int_{t_0}^{t} u_1(x_0^+, z_0^+, t') dt' = \frac{2U_0 \cos(k_z z_0^+)}{\omega} (\cos \phi_0(t) - \cos \phi_0(0))
\]

(B3)

and

\[
\Delta z^+ \approx \int_{t_0}^{t} w_1(x_0^+, z_0^+, t') dt' = \frac{2U_0 \cot \theta \sin(k_z z_0^+)}{\omega} (\sin \phi_0(t)) - \sin \phi_0(0),
\]

(B4)

where \(\phi_0(t) \equiv \phi(x_0^+, t)\) and \(\phi_0(0) \equiv \phi(x_0^+, t_0)\). The gradients of \(u\) at \((x_0^+, z_0^+)\), which are approximated up to \(O(A)\) by those of \(u_1\), are

\[
\left. \frac{\partial u}{\partial x} \right|_{(x_0^+, z_0^+, t)} \approx \left. \frac{\partial u_1}{\partial x} \right|_{(x_0^+, z_0^+, t)} = 2k_x U_0 \cos(k_z z_0^+) \cos \phi_0(t),
\]

(B5)

\[
\left. \frac{\partial u}{\partial z} \right|_{(x_0^+, z_0^+, t)} \approx \left. \frac{\partial u_1}{\partial z} \right|_{(x_0^+, z_0^+, t)} = -2k_x U_0 \sin(k_z z_0^+) \sin \phi_0(t).
\]

(B6)

The Stokes operator in (A10) can now be readily computed to obtain the two contributors to Stokes drift

\[
\langle \frac{\partial u}{\partial x} \Delta x^+ \rangle \approx \langle \frac{\partial u_1}{\partial x} \int u_1 dt \rangle = \frac{2k_x U_0^2 \cos^2(k_z z_0^+)}{\omega} \approx \frac{2U_0^2}{c_x} \cos^2(k_z z_0^+).
\]

(B7)
and
\[
\left\langle \frac{\partial u}{\partial z} \Delta z^t \right\rangle \approx \left\langle \frac{\partial u}{\partial z} \right\rangle \int w_I dt = -\frac{2}{\omega} k_z U_0^2 \sin^2(k_z z_0^t) = -\frac{2 U_0^2}{c_x} \sin^2(k_z z_0^t),
\]
which sum up to the total \(O(A^2)\) Stokes drift
\[
u_{S,II} = \frac{2 U_0^2}{c_x} (\cos^2(k_z z_0^t) - \sin^2(k_z z_0^t)) = \frac{2 U_0^2}{c_x} \cos(2k_z z_0^t)
\]
due to the \(O(A)\) primary wave. The \(O(A^2)\) Stokes drift \(\nu_{S,II}\) has dependence on neither \(x\), a homogeneous direction in the setup, nor \(t_0\), the initial time of the wave averaging.

3. Euler-Stokes cancelation

With Eqs. (B1), (B9), and (A13), one can obtain a second-order accurate estimate of the mean Lagrangian drift,
\[
\langle u_{L,II} \rangle = \langle u \rangle + \nu_{S,II} = \frac{2 U_0^2}{c_x} \cos(2k_z z_0^t) = 0,
\]
which suggests that the second-order Eulerian mean flow due to wave-wave interaction and Stokes drift due to leading-order primary wave cancel out each other completely within a reflecting plane IW, yielding zero Lagrangian mean flow up to \(O(A^2)\). To reach this conclusion, the inviscid solutions to the reflection problem have been used and small-amplitude approximations invoked.


28. Here and elsewhere in this paper, the definition of $A$ differs from that was used in the work of Z&D (2013). It can be shown that $A = 2\pi \tan \theta A_{2D}$, where $A_{2D}$ is the wave amplitude parameter defined by Eq. (3) of Z&D.

29. Here and elsewhere, roman numerals in subscripts denote orders of dynamics, e.g., “II” in the subscript of $\nu_{S, II}$ denotes an $O(A^2)$ effect.


50. G. Gil and O. B. Fringer, personal communication.

51. L. Gostiaux, personal communication.